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Holographic Duality, Supersymmetry, and Painlevé Equation

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Abstract

A new holographic duality between four-dimensional gravity and two-dimensional quantum field theories is found. The constraints on the target space geometry of the four-dimensional N=2 supersymmetric non-linear sigma-models in N=2 supergravity background are interpreted as the Renormalization Group (RG) flow equations in the two-dimensional N=2 supersymmetric quantum field theories. Our geometrical description of the RG flow is manifestly covariant. The proposed holography is based on the $SU(2)$-invariant (Tod-Hitchin) exact solutions describing (Weyl) anti-self-dual Einstein metrics, governed by the (integrable) Painlevé VI equation. The regular metric solutions in the bulk are parametrized by the 'boundary' central charge, whereas the coefficients of the RG flow equations are universal, being given by the OPE coefficients of an N=2 superconformal field theory on the boundary.

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1 Introduction

The AdS/CFT correspondence in its original formulation [1] relates the type-IIB superstring theory on $AdS_5 \times S^5$ with the four-dimensional (4d), N=4 supersymmetric $SU(N)$ Yang-Mills theory on the boundary of the AdS_5 space. The string loop corrections turn out to be proportional to N^{-2}, whereas the α'-corrections are proportional to $\lambda^{-1/2}$, where $\lambda = g^2_{\rm YM} N$ is the 't Hooft coupling, so that the large-N and large-λ (strong coupling) limit can be investigated in the AdS-supergravity approximation (see ref. [2] for a review).

Being conformally invariant, the quantum 4d, N=4 super-Yang-Mills (SYM) theory is believed to be self-dual with respect to the electric-magnetic duality. A massive deformation of the N=4 SYM theory breaks down both N=4 supersymmetry and conformal invariance, which result in a non-trivial Renormalization Group (RG) flow. The radial coordinate of the AdS_5 gives the natural scale to the gauge theory. The so-called *holographic duality* means an identification of the supergravity equations of motion in the bulk with the RG-flow equations in the gauge theory on the boundary [3].

The holographic prescription for computing Green's functions [1, 2] may still be valid far beyond the original (superconformal) Maldacena proposal, so that it could be applied to any case where a gravity in the bulk can be related to a Quantum Field Theory (QFT) on the boundary. To test this more general proposal, it is worthy to find the examples of a holographic correspondence in *lower* dimensions, where both the QFT strong coupling and the exact gravity solutions are well-understood, being under control. In this Letter we give such an example.

We show the existence of a holographic correspondence between classical *four-dimensional* gravity and *two-dimensional* QFT. A new type of the holographic duality is found by identifying the geometrical constraints on the target space geometry of the Non-Linear Sigma-Models (NLSM), in the four-dimensional N=2 supergravity background, with the RG-flow equations in the two-dimensional N=2 supersymmetric QFT. For simplicity, we restrict ourselves to the four-dimensional NLSM target spaces, by taking a single hypermultiplet coupled to N=2 supergravity in four dimensions. This limited class of 4d NLSM corresponds to intergrable (non-conformal) deformations of the two-dimensional N=2 Superconformal Field Theories (SCFT) whose chiral ring is of dimension three. The proposed duality heavily relies on the mechanism assigning a two-dimensional conformal boundary to the four-dimensional, conformally anti-self-dual Einstein spaces of negative scalar curvature. The 4d, N=2 NLSM metric at a fixed point is conformally equivalent to the Zamolodchikov metric of the 2d, N=2 superconformal QFT [4]. The simplest example of the four-dimensional Bergmann sphere appears to be related to the 2d, N=4 superconformal field theory on its two-dimensional boundary. The off-critical solutions, describing the RG flow, are highly constrained by supersymmetry, while they are governed by the solutions to the Painlevé VI equation. It is demonstrated that the celebrated exact metric solutions of Tod [5] and Hitchin [6] have the natural interpretation in the context of holographic duality.

2 UV fixed-point correspondence

As was noticed many years ago by Bagger and Witten [7], the N=2 scalar (hypermultiplet) couplings in the four-dimensional N=2 supergravity are described by the NLSM with the quaternionic-Kähler target spaces of negative scalar curvature. The four-dimensional quaternionic-Kähler target space is (Weyl) Anti-Self-Dual (ASD) and Einstein [8], i.e. the NLSM metric should obey the equations

$$W^{+}_{abcd} = 0 \quad \text{and} \quad R_{ab} = \frac{1}{2}\Lambda g_{ab} \ , \qquad \Lambda = -24\kappa^2 \ , \tag{1}$$

where $W = W^{-} + W^{+}$ is the Weyl tensor, R_{ab} is the Ricci tensor, $a, b, c, d = 1, 2, 3, 4$, and κ is the gravitational coupling constant.

Given a simple Lie group G, the associated quaternionic *symmetric* space is unique, and it is called the Wolf space [9],

$$\frac{G}{H_{\perp} \times SU(2)_{\chi}} \ , \tag{2}$$

where χ is the highest root of G, the $SU(2)_{\chi}$ is the subalgebra of G, associated with the root ψ, and $H_{\perp}$ is the centralizer of $SU(2)_{\chi}$ in G. In four dimensions, there are only two Wolf cosets of negative scalar curvature, $SO(4,1)/SO(4)$ and $SU(2,1)/U(2)$. Since all Wolf spaces have the $SU(2)$ isometry, which is going to play the crucial role in our investigation, the natural metrics in those spaces are conveniently described (like in general relativity) by the Bianchi IX formalism with manifest $SU(2)$ symmetry. Given a 'radial' coordinate r and 'Euler angles' (θ, ψ, ϕ), the $SU(2)$-covariant one-forms, $\sigma_1 = \frac{1}{2}(\sin\psi d\theta - \sin\theta\cos\psi d\phi)$, $\sigma_2 = -\frac{1}{2}(\cos\psi d\theta + \sin\theta\sin\psi d\phi)$ and $\sigma_3 = \frac{1}{2}(d\psi + \cos\theta d\phi)$, are subject to the relation $\sigma_i \wedge \sigma_j = \frac{1}{2}\varepsilon_{ijk} d\sigma_k$. The natural metric, associated with the symmetric (Euclidean AdS_4) space $SO(4,1)/SO(4)$, is conformally flat,

$$ds^2 = \frac{1}{(1-r^2)^2}\left[dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)\right] \ . \tag{3}$$

The boundary ($r \to 1^-$) is given by a three-dimensional sphere S^3, while the four-dimensional conformal strucrure in the ball $r^2 < 1$ induces the conformal structure on S^3. However, we are going to exclude this case since the gauge theories in three dimensions are not conformally invariant, while there is no conformal anomaly at all (the latter is true in any odd-dimensional space).

The remaining symmetric Wolf space $SU(2,1)/U(2)$ is perfectly suitable for our purposes. The natural metric in this space is given by the Bergmann metric (dual to the Fubini-Study metric) [8],

$$ds^2 = \frac{dr^2}{(1-r^2)^2} + \frac{r^2}{(1-r^2)^2}\sigma_2^2 + \frac{r^2}{(1-r^2)}(\sigma_1^2 + \sigma_3^2) \ . \tag{4}$$

The conformal structure, associated with the metric (4) inside the unit ball in $\mathbf{C}^2$, does not extend across the boundary since the coefficient at σ_2^2 decays *faster* than the coefficients at σ_1^2 and σ_3^2. The conformal structure, however, survives on the *two-dimensional* (2d) subspace of S^3, which is annihilated by σ_2, since it is protected by the Kähler nature of the metric (4).

The symmetric space $SU(2,1)/U(1) \times SU(2)$ is related to the UV fixed point of a 2d QFT. It is not difficult to identify the corresponding SCFT in 2d, because all Wolf spaces appear in the Kazama-Suzuki coset construction list (see ref. [10] for a review). The Wolf spaces are, in fact, naturally associated with the 2d, N=4 superconformal symmetry, albeit non-linearly realized [11]. The formal central charge of the 2d SCFT, associated with $SU(2,1)/U(1) \times SU(2)$, is given by $c = 3(3p+1)/(p+3)$ [11]. Note that $c \to 9$ when $|p| \to \infty$.

The type-IIA superstring compactification on Calabi-Yau spaces is well-known to assign the dilaton to the hypermultiplet that also contains the NS-NS axion and the complex Ramond-Ramond-type scalar. The hypermultiplet moduli space metric obeys eq. (1), while at the tree string level one finds the quaternionic manifold that is similar to $SU(2,1)/U(1) \times SU(2)$ but has the Heisenberg symmetry group instead of $SU(2)$ [12].

3 Self-duality and RG flow

The solutions to eq. (1), which can be interpreted as the RG flow, are supposed to share the most basic features of the latter: (i) they are to obey the first-order differential equations, and (ii) there should be a well-defined RG flow parameter. The latter naturally implies the $SU(2)$ isometry of the metric because the (non-degenerate) action of this isometry leads to the three-dimensional orbits that can be parametrized by the 'radial' coordinate (t) to be identified with the RG scale. The (Weyl) ASD equations on the metric then take the form of a first-order system of Ordinary Differential Equations (ODE), so that the feature (i) is automatic. In other words, in the context of the holographic duality, we should add the $SU(2)$ symmetry to the general requirements (1) dictated by four-dimensional local N=2 supersymmetry alone. In fact, the $SU(2)$ symmetry can also be considered as the part of N=2 supersymmetry, after being identified with the automorphism group of the rigid N=2 supersymmetry algebra (R-symmetry).

We are thus led to a study of the $SU(2)$-invariant deformations of the Bergmann metric (4) subject to the constraints (1) with the quaternionic-Kähler property. This problem was already addressed on another occasion by Tod [5] and Nitchin [6]. A generic $SU(2)$ invariant metric in the Bianchi IX formalism reads

$$ds^2 = w_1 w_2 w_3 dt^2 + \frac{w_2 w_3}{w_1}\sigma_1^2 + \frac{w_3 w_1}{w_2}\sigma_2^2 + \frac{w_1 w_2}{w_3}\sigma_3^2 \ , \tag{5}$$

where we have taken it in the diagonal form without a loss of generality. As was demonstrated by Tod [5], the Weyl ASD conditions of eq. (1) result in the classical Halphen system [13]

$$\dot{A}_1 = -A_2 A_3 + A_1(A_2 + A_3) \quad \text{and cyclic permutations} \ , \tag{6}$$

where a dot means differentiation with respect to t, and the A_i, $i = 1, 2, 3$, are defined from the system

$$\dot{w}_1 = -w_2 w_3 + w_1(A_2 + A_3) \quad \text{and cyclic permutations} \ . \tag{7}$$

The Bergmann metric corresponds to the case when all A_i vanish. The Einstein condition in eq. (1) can be satisfied after a conformal rescaling of the (Weyl) ASD metric (see below). Having solved eq. (6), the solution can be substituted into eq. (7). To solve the last equations, it is convenient to change variables as [5]

$$w_1 = \frac{\Omega_1 \dot{x}}{\sqrt{x(1-x)}} \ , \quad w_2 = \frac{\Omega_2 \dot{x}}{\sqrt{x^2(1-x)}} \ , \quad w_3 = \frac{\Omega_3 \dot{x}}{\sqrt{x(1-x)^2}} \ , \tag{8}$$

where Ω_i are constrained by

$$\Omega_2^2 + \Omega_3^2 - \Omega_1^2 = \tfrac{1}{4} \ . \tag{9}$$

Equation (7) then takes the form [5, 6]

$$\Omega_1' = -\frac{\Omega_2 \Omega_3}{x(1-x)} \ , \quad \Omega_2' = -\frac{\Omega_3 \Omega_1}{x} \ , \quad \Omega_3' = -\frac{\Omega_1 \Omega_2}{1-x} \ , \tag{10}$$

where prime denotes differentiation with respect to x. The constraint (9) is preserved under eq. (10), so that the transformation (8) is consistent. In terms of the new variables (x, Ω_i), the Einstein condition (1) on the metric

$$ds^2 = e^{2u}\left[\frac{dx^2}{x(1-x)} + \frac{\sigma_1^2}{\Omega_1^2} + \frac{(1-x)\sigma_2^2}{\Omega_2^2} + \frac{x\sigma_3^2}{\Omega_3^2}\right] \ , \tag{11}$$

amounts to the relation [5]

$$96\kappa^2 e^{2u} = \frac{8x\Omega_1^2\Omega_2^2\Omega_3^2 + 2\Omega_1\Omega_2\Omega_3(x(\Omega_1^2+\Omega_2^2) - (1-4\Omega_3^2)(\Omega_2^2-(1-x)\Omega_1^2))}{(x\Omega_1\Omega_2 + 2\Omega_3(\Omega_2^2-(1-x)\Omega_1^2))^2} \ . \tag{12}$$

The ODE system (6), $\dot{A}_i = C_i^{jk} A_j A_k$, can be naturally interpreted as the RG flow equations in the (non-conformal) QFT originating from the conformal field theory on the two-dimensional boundary, by assuming that the corresponding metric solutions have the asymptotical behaviour similar to that of the Bermann metric, when the metric coefficient at σ_2^2 in eq. (5) decays faster than the others (see the next sect. 4). The Kähler structure in the bulk should also be extendable to the two-dimensional conformal boundary annihilated by σ_2. The Kähler nature of the two-dimensional conformal field theory implies that it should be N=2 supersymmetric. Therefore, the Zamolodchikov metric [4] in the quantum moduli space of the N=2 SCFT on the boundary should be a Kähler metric. Putting everything together, we thus arrive at the N=2 supersymmetric QFT that can be thought of as the deformation of the 2d, N=2 SCFT governed by the RG flow equation (6) with t as the deformation parameter.

The most natural (minimal) two-dimensional N=2 SCFT models with the Kähler (Zamolodchikov) metric are naturally described by an N=2 chiral (Landau-Ginzburg) potential [10]. The proposed physical (holographic) interpretation of the exact metric solution to eq. (1) requires the metric to be regular (or complete) inside of its range of definition, so that all of its pole singularities (in a particular parametrization) are to be removable by coordinate transformations.

The most fundamental feature of any 2d, N=2 SCFT is the existence of a chiral ring of primary operators, governed by a holomorphic potential [10]. The integrable N=2 supersymmetric QFT are obtained from the N=2 minimal models perturbed by the so-called relevant operators built out of the chiral primaries [14]. [2] This implies that the coefficients C_i^{jk} in eq. (6) are universal, while they can be identified with the OPE (or fusion) coefficients of the chiral ring.

[2]By the 'solvable' (or integrable) two-dimensional QFT we mean the 2d QFT with a factorizable S-matrix, in the sense of Zamolodchikov [15].

4 Painlevé VI equation and complete solution

The ODE system (6) has a long history [16]. Perhaps, its most natural interpretation is provided via a reduction of the $SL(2,\mathbf{C})$ anti-self-dual Yang-Mills theory from four to one dimension [17]. A classification of all possible reductions is known in terms of the so-called *Painlevé* groups that give rise to six different types of integrable Painlevé equations [17]. It remains to identify those of them that lay behind the Weyl-ASD quaternionic-Kähler geometry with $SU(2)$ symmetry. There are only two natural (or nilpotent, in the terminology of ref. [17]) types (III and VI) that give rise to a single non-linear integrable equation. In the geometrical terms, it is the Painlevé III equation that lays behind the four-dimensional Kähler spaces with vanishing scalar curvature [18], whereas the Painlevé VI equation is known to lay behind the Weyl-ASD geometries having the $SU(2)$ symmetry [5, 6, 19]. A generic Painlevé VI equation has four real parameters [17], but they are all fixed by the quaternionic-Kähler property [5, 6]. It results in the particular Painlevé VI equation

$$\begin{aligned} y'' &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)(y')^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y' \\ &\quad + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\frac{1}{8}-\frac{x}{8y^2}+\frac{x-1}{8(y-1)^2}+\frac{3x(x-1)}{8(y-x)^2}\right] , \end{aligned} \tag{13}$$

where $y=y(x)$, and prime denotes differentiation with respect to x.

The equivalence between eqs. (6) and (13) via eq. (10) is well-known to mathematicians [5, 6, 19]. Explicitly, in the Einstein case, it reads

$$\begin{aligned} \Omega_1^2 &= \frac{(y-x)^2y(y-1)}{x(1-x)}\left(v-\frac{1}{2(y-1)}\right)\left(v-\frac{1}{2y}\right) , \\ \Omega_2^2 &= \frac{(y-x)y^2(y-1)}{x}\left(v-\frac{1}{2(y-x)}\right)\left(v-\frac{1}{2(y-1)}\right) , \\ \Omega_3^2 &= \frac{(y-x)y(y-1)^2}{(1-x)}\left(v-\frac{1}{2y}\right)\left(v-\frac{1}{2(y-x)}\right) , \end{aligned} \tag{14}$$

where the auxiliary variable v is defined by the equation

$$y' = \frac{y(y-1)(y-x)}{x(x-1)}\left(2v-\frac{1}{2y}-\frac{1}{2(y-1)}+\frac{1}{2(y-x)}\right) . \tag{15}$$

A solution to eq. (13), leading to a *complete* (regular) metric, is known to be unique, while it can be written down in terms of the theta-functions $\vartheta_\alpha(z|\tau)$ where

$\alpha = 1,2,3,4.$ [3] One finds [21, 6]

$$
\begin{aligned}
y(x) \; &= \; \frac{\vartheta_1'''(0)}{3\pi^2\vartheta_4^4(0)\vartheta_1'(0)} + \frac{1}{3}\left[1+\frac{\vartheta_3^4(0)}{\vartheta_4^4(0)}\right] \\
&+ \frac{\vartheta_1'''(z)\vartheta_1(z) - 2\vartheta_1''(z)\vartheta_1'(z) + 2\pi i(\vartheta_1''(z)\vartheta_1(z) - \vartheta_1'^2(z))}{2\pi^2\vartheta_4^4(0)\vartheta_1(z)(\vartheta_1'(z)+\pi i\vartheta_1(z))} \; ,
\end{aligned}
\tag{16}
$$

where $x = \vartheta_3^4(0)/\vartheta_4^4(0)$, $z = \frac{1}{2}(\tau - k)$, and $k > 0$ is an arbitrary real parameter describing the monodromy of the solution (16) around its essential singularities (branch points) $x = 0,1,\infty$. This (non-abelian) monodromy is generated by the matrices (with the eigenvalues $\pm i$) [6]

$$
M_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \; , \quad M_2 = \begin{pmatrix} 0 & i^{1-k} \\ i^{1+k} & 0 \end{pmatrix} \; , \quad M_3 = \begin{pmatrix} 0 & i^{-k} \\ -i^k & 0 \end{pmatrix} \; . \tag{17}
$$

Yet another (equivalent) form of a solution to the metric coefficients w_i in eq. (7) was found in ref. [22], in terms of the theta functions with characteristics, by the use of the related Schlesinger system and the isomonodromic deformation techniques.

The function (16) is meromorphic outside $x = 0,1,\infty$, with the simple poles at $\bar{x}_1, \bar{x}_2, \ldots$, where $\bar{x}_n \in (x_n, x_{n+1})$ and $x_n = x(ik/(2n-1))$ for each positive integer n. Accordingly, the metric is well-defined (complete) for $x \in (\bar{x}_n, x_{n+1}]$, i.e. in the unit ball with the origin at $x = x_{n+1}$ and the boundary at $x = \bar{x}_n$ [6]. Near the boundary the metric (11) has the asymptotical behaviour [6]

$$
\begin{aligned}
ds^2 \; &= \; \frac{dx^2}{(1-x)^2} + \frac{4}{(1-x)\cosh^2(\pi k/2)}\sigma_1^2 + \frac{16}{(1-x)^2\sinh^2(\pi k/2)\cosh^2(\pi k/2)}\sigma_2^2 \\
&+ \frac{4}{(1-x)\sinh^2(\pi k/2)}\sigma_3^2 + \quad \text{regular terms} \; .
\end{aligned}
\tag{18}
$$

It is clear from eq. (18) that the coefficient at σ_2^2 vanishes faster than the others, like in eq. (4), so that there is the natural conformal structure,

$$
\sinh^2(\pi k/2)\sigma_1^2 + \cosh^2(\pi k/2)\sigma_3^2 \; , \tag{19}
$$

on the two-dimensional boundary annihilated by σ_2. The only relevant parameter $\tanh^2(\pi k/2)$ of eq. (19) is essentially the central charge of the two-dimensional conformal field theory on the boundary. In the interior of the ball we have the spectral flow, with the 'effective' central charge (c-function) being monotonically decreasing according to the c-theorem [4]. The RG evolution ends at another (IR) fixed point where the solution (16) has a removable pole.

[3] We use the standard definitions and notation for the theta functions from ref. [20].

5 Conclusion

The proposed holographic duality maps fixed points into fixed points, and gives the simple and natural interpretation of the 2d RG-flow in the context of 4d gravity. Local 4d, N=2 supersymmetry appears to be the sole source of the fundamental constraints (1) on the metric. The regular solution to the metric is also unique, being parametrized by the central charge of the conformal field theory on the boundary. It is also worth noticing that our geometrical description of the RG flow by eq. (1) is manifestly covariant — *cf.* ref. [23]. Since the coefficients of the ODE system (6) are universal, being essentially determined by the chiral ring, the whole metric governing the RG flow is apparently dictated by the chiral ring of the N=2 SCFT on the boundary, i.e. by topology only ! In the case of the N=2 minimal models, the chiral ring is dictated by a Landau-Ginzburg potential. It may be possible to generalize our example of the holographic duality to the multi-dimensional quaternionic-Kähler metrics of negative scalar curvature, by relating them to the RG flow in two-dimensional QFT to be obtained by integrable (and N=2 supersymmetric) deformations of the N=2 *rational* SCFT.

The integrable deformations of the 2d, N=2 superconformal Landau-Ginzburg models by the most relevant operators were analyzed by Cecotti and Vafa [24]. They found that the ground state metric satisfies classical Toda equations whose solutions are governed by the Painlevé III equation in a series of examples. From the viewpoint of our holographic duality, the Cecotti-Vafa solutions are naturally associated with the Kähler metrics of vanishing scalar curvature [18] when the background gravity decouples, $\kappa = 0$.

The constraints (1) do not seem to imply any quantization condition on the monodromy parameter k since the regular metric solutions exist for any $k > 0$ [6]. However, the related central charge $c > 0$ on the boundary is naturally quantized (and bounded from above) for the minimal N=2 superconformal models associated with compact (simply-laced) Lie groups. A resolution of this puzzle may be related to the *negative* curvature of the metrics. The ASD Einstein metrics of *positive* curvature take the similar form (11), while they are known to be related to Poncelet n-polygons, which implies the quantization condition $k = 2/n$, where $n \in \mathbf{Z}$ [25]. So, it seems that we are dealing with the 'boundary' conformal field theories based on non-compact groups. Perhaps, the Tod-Hitchin metrics may also be interpreted as the kink-type gravitational solitons preserving some supersymmetry in the context of higher-dimensional supergravity (*cf.* ref. [3]). It would be interesting to investigate possible connections to matrix models, 2d gravity and non-commutative geometry.

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